Algebraic Number Theory

Exercise Sheet 9

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Exercise 1. Let A be a Dedekind ring. Let k be a positive integer. For $i = 1, ..., k$ let $\mathcal{P}_i$ be distinct prime ideals of A, x_i elements of A, and n_i positive integers. Show that there exists an element $x \in A$, such that $\upsilon_{\mathcal{P}_i}(x - x_i) \geq n_i$ for every $i = 1, ..., k$.
Hint: Start with the case: $x_1 \in A$ and $x_2 = ... = x_k = 0$. Consider the ideal $\mathcal{P}_1^{n_1} + \mathcal{P}_2^{n_2} ... \mathcal{P}_k^{n_k}$.

Exercise 2. Show that a Dedekind ring with only finitely many prime ideals is a principal ideal domain.
Hint: Use Exercise 1.

Exercise 3. Let d be a square-free integer. Let $K = \mathbb{Q}(\alpha)$ be a quadratic field, where $\alpha^2 = d$.
(1) Show that $\mathcal{O}_K$ is principal, if $d = 2, 5, -11, 7$.
(2) Find $C(\mathcal{O}_K)$ for $d = -6$.
Hint for (1) and (2): Use the inequality from Korollar 9 (Chapter II) and estimate $(\frac{4}{\pi})^{r_2} \frac{n!}{n^n} \sqrt{|d_K|}$ in every case.